Exercise 75

Show that the function $y = e^{2x}(A\cos 3x + B\sin 3x)$ satisfies the differential equation $y'' - 4y' + 13y = 0$.

Solution

Differentiate the given function using the product rule and the chain rule.

$$\begin{aligned}
y' &= \frac{dy}{dx} \\
&= \frac{d}{dx}[e^{2x}(A\cos 3x + B\sin 3x)] \\
&= \left[\frac{d}{dx}(e^{2x})\right](A\cos 3x + B\sin 3x) + e^{2x}\left[\frac{d}{dx}(A\cos 3x + B\sin 3x)\right] \\
&= \left[e^{2x}\cdot\frac{d}{dx}(2x)\right](A\cos 3x + B\sin 3x) + e^{2x}\left[A\frac{d}{dx}(\cos 3x) + B\frac{d}{dx}(\sin 3x)\right] \\
&= \left[e^{2x}\cdot(2)\right](A\cos 3x + B\sin 3x) + e^{2x}\left[A(-\sin 3x)\cdot\frac{d}{dx}(3x) + B(\cos 3x)\cdot\frac{d}{dx}(3x)\right] \\
&= 2e^{2x}(A\cos 3x + B\sin 3x) + e^{2x}\left[A(-\sin 3x)\cdot(3) + B(\cos 3x)\cdot(3)\right] \\
&= e^{2x}[(2A + 3B)\cos 3x + (-3A + 2B)\sin 3x]
\end{aligned}$$

Take another derivative.

$$\begin{aligned}
y'' &= \frac{d}{dx}(y') \\
&= \frac{d}{dx}\{e^{2x}[(2A + 3B)\cos 3x + (-3A + 2B)\sin 3x]\} \\
&= \left[\frac{d}{dx}(e^{2x})\right][(2A + 3B)\cos 3x + (-3A + 2B)\sin 3x] + e^{2x}\left\{\frac{d}{dx}[(2A + 3B)\cos 3x + (-3A + 2B)\sin 3x]\right\} \\
&= (2e^{2x})[(2A + 3B)\cos 3x + (-3A + 2B)\sin 3x] + e^{2x}[(2A + 3B)(-3\sin 3x) + (-3A + 2B)(3\cos 3x)] \\
&= e^{2x}[(4A + 6B - 9A + 6B)\cos 3x + (-6A + 4B - 6A - 9B)\sin 3x] \\
&= e^{2x}[(-5A + 12B)\cos 3x + (-12A - 5B)\sin 3x]
\end{aligned}$$

Now plug these formulas into the differential equation and check to see if the left side evaluates to zero.

$$\begin{aligned}
y'' - 4y' + 13y &= e^{2x}[(-5A + 12B)\cos 3x + (-12A - 5B)\sin 3x] \\
&\qquad - 4e^{2x}[(2A + 3B)\cos 3x + (-3A + 2B)\sin 3x] \\
&\qquad\qquad + 13e^{2x}(A\cos 3x + B\sin 3x) \\
&= e^{2x}[(-5A - 8A + 13A + 12B - 12B)\cos 3x + (-12A + 12A - 5B - 8B + 13B)\sin 3x] \\
&= 0
\end{aligned}$$